On Cui-Kano's Characterization Problem on Graph Factors

H. L. Lu[1] and David G. L. Wang[2]

[1]Department of Mathematics
Xi'an Jiaotong University, Xi'an 710049, P. R. China
[1]luhongliang215@sina.com

[2]Beijing International Center for Mathematical Research
Peking University, Beijing 100871, P. R. China
[2]wgl@math.pku.edu.cn

Abstract

An H_n-factor of a graph G is defined to be a spanning subgraph F of G such that each vertex has degree belonging to the set $\{1, 3, 5, \ldots, 2n-1, 2n\}$ in F. In this paper, we investigate H_n-factors of graphs by using Lovász's structural descriptions to the degree prescribed subgraph problem. We find some sufficient conditions for the existence of an H_n-factor of a graph. In particular, we make progress on the characterization problem for a special family of graphs proposed by Cui and Kano in 1988.

Keywords: Lovász's structural description, H_n-factor, H_n-decomposition

2010 AMS Classification: 05C75

1 Introduction

Let G be a simple graph with vertex set $V(G)$. Let $f, g\colon V(G) \to \mathbb{Z}$ be functions on $V(G)$. A (g, f)-factor of G is a spanning subgraph F such that

$$g(v) \le d_F(v) \le f(v)$$

for any vertex v, where $d_F(v)$ is the degree of v in F. In particular, if there exist integers a and b such that $g(v) = a$ and $f(v) = b$ for all vertices v, then the (g, f)-factor is called an $[a, b]$-factor. For example, connected $[2, 2]$-factors are nothing but Hamiltonian cycles, and $[1, 1]$-factors are perfect matchings. There is a large amount of literature on graph factors, see Plummer [10], Liu and Yu [12], and Akiyama and Kano [1] for surveys. For connected factors, we refer the reader to Kouider and Vestergaard [6].

Let H be a function associating a subset of $\mathbb{Z}$ with each vertex of G, called a degree prescription. It is natural to generalize (g, f)-factors to H-factors, i.e., spanning subgraphs F such that

$$d_F(v) \in H(v) \tag{1.1}$$

for all vertices v. Let F be a spanning subgraph of G. Following Lovász [9], one may measure the "deviation" of F from the condition (1.1) by

$$\delta_H(F) = \sum_{v \in V(G)} \min\bigl\{|d_F(v) - h| :\ h \in H(v)\bigr\}. \tag{1.2}$$

Moreover, the "solvability" of (1.1) can be characterized by

$$\delta(H) = \min\{\delta_H(F) :\ F \text{ is a spanning subgraph of } G\}.$$

The subgraph F is said to be H-optimal if $\delta_H(F) = \delta(H)$. It is clear that F is an H-factor if and only if $\delta_H(F) = 0$, and any H-factor (if exists) is H-optimal.

In [8], Lovász proposed the problem of determining the value of $\delta(H)$, called the degree prescribed subgraph problem. Let

$$H = \{h_1, h_2, \ldots, h_n\}$$

be a set of integers, where $h_1 < h_2 < \cdots < h_n$. It is said to be an allowed set if each of its gaps has at most one integer, i.e.,

$$h_{i+1} - h_i \le 2, \quad \forall\, 1 \le i \le n-1.$$

We say that a prescription H is allowed if $H(v)$ is an allowed set for all vertices v. Lovász [9] built up a whole theory to the degree prescribed subgraph problem in case that H is an allowed prescription. He showed that the problem is NP-complete without the restriction that H is allowed. Cornuéjols [3] provided the first polynomial algorithm for the problem with H allowed.

A special case of the degree prescribed subgraph problem is the so-called f-parity subgraph problem, i.e., the problem with

$$H(v) = \{\ldots,\ f(v) - 4,\ f(v) - 2,\ f(v)\}$$

for some function $f\colon V(G) \to \mathbb{Z}$. The first investigation of the f-parity subgraph problem is due to Amahashi [2], who gave a Tutte type characterization for graphs having a global odd factor. Let S be a subset of $V(G)$. Denote by $G - S$ the subgraph of G obtained by removing all vertices in S. Denote by $o(G)$ the number of odd components of G. Let $n \ge 2$ be an integer independent of the number of vertices of G. Let H_o be the prescription associating the first n positive odd integers with each vertex, i.e.,

$$H_o(v) = \{1, 3, 5, \ldots, 2n-1\}.$$

Theorem 1.1 (Amahashi) *A graph G has an H_o-factor if and only if*

$$o(G - S) \le (2n-1)\,|S|, \quad \forall\, S \subseteq V(G). \tag{1.3}$$

For general odd value functions f, Cui and Kano [4] established a Tutte type theorem. Noticing the form of the condition (1.3), they asked the question of characterizing graphs G in terms of graph factors such that

$$o(G-S) \le 2n\,|S|, \quad \forall\, S \subseteq V(G). \tag{1.4}$$

For more studies on the f-parity subgraph problem, see Topp and Vestergaard [11], and Kano, Katona, and Szabó [5].

Motivating by solving Cui-Kano's problem, we consider the degree prescribed subgraph problem for the special prescription

$$H_n(v) = H_o(v) \cup \{2n\} = \{1, 3, 5, \ldots, 2n-1, 2n\}. \tag{1.5}$$

We shall study the structure of graphs which have no H_n-factors by using Lovász's theory [9]. Consequently, we obtain that any graph satisfying the condition (1.4) contains an H_n-factor.

Besides many applications of Lovász's structural description for special families of graphs and special allowed prescriptions, there is much attention paid to finding sufficient conditions for the existence of an H-factor in a graph for special prescriptions H, see [10]. In this paper, we also give some sufficient conditions for the existence of an H_n-factor in a graph.

2 The main result

In this section, we study H_n-factors of graphs based on Lovász's structural description of the degree prescribed subgraph problem.

Let H be an allowed prescription. Denote by $I_H(v)$ the set of vertex degrees in all H-optimal subgraphs, i.e.,

$$I_H(v) = \{d_F(v) \mid F \text{ is } H\text{-optimal}\}.$$

Comparing the set $I_H(v)$ with $H(v)$, one may partition the vertex set $V(G)$ into four classes:

$$\begin{aligned}
C_H &= \{v \in V(G) :\ I_H(v) \subseteq H(v)\},\\
A_H &= \{v \in V(G)\backslash C_H :\ \min I_H(v) \ge \max H(v)\},\\
B_H &= \{v \in V(G)\backslash C_H :\ \max I_H(v) \le \min H(v)\},\\
D_H &= V(G)\backslash A_H \backslash B_H \backslash C_H.
\end{aligned}$$

It is clear that the 4-tuple (A_H, B_H, C_H, D_H) is a pairwise disjoint partition of $V(G)$. We call it the H-decomposition of G. In fact, the four subsets can be distinguished

according to the contributions of their members to the deviation (1.2). A graph G is said to be H-critical if it is connected and $D_H = V(G)$.

In [9, Theorem (2.1)], Lovász gave the following property for the subset D_H.

Lemma 2.1 (Lovász) *If $D_H \neq \emptyset$, then the intersection*

$$[\min I_H(v),\, \max I_H(v)] \cap H(v)$$

contains no consecutive integers for any vertex $v \in D_H$.

In [9, Corollary (2.4)], Lovász gave the following structural result.

Lemma 2.2 (Lovász) *There is no edge between C_H and D_H.*

For any subset $S \subseteq V(G)$, denote by $G[S]$ the subgraph induced by S. Denote the number of components of $G[S]$ by $c(S)$, and the number of odd components of $G[S]$ by $o(S)$. In [9, Theorem (4.3)], Lovász established the formula

$$\delta(H) = c(D_H) + \sum_{v \in B_H} \min H(v) - \sum_{v \in A_H} \max H(v) - \sum_{v \in B_H} d_{G-A_H}(v).$$

By definition, G contains no H-factors if and only if $\delta(H) > 0$. This yields the next lemma immediately.

Lemma 2.3 (Lovász) *A graph G contains no H-factors if and only if*

$$c(D_H) + \sum_{v \in B_H} \min H(v) > \sum_{v \in A_H} \max H(v) + \sum_{v \in B_H} d_{G-A_H}(v). \tag{2.1}$$

Let $X \subseteq V(G)$. For any vertex v, define

$$H_X(v) = \{h - e(v, X) :\ h \in H(v)\}, \tag{2.2}$$

where $e(v, X)$ is the number of edges from v to X. In [9, Theorem (4.2)], Lovász showed that each component of $G[D_H]$ is H'-critical where $H' = H_{B_H}$. He [9, Lemma (4.1)] also obtained that $\delta(H) = 1$ if G is H-critical. This leads to the next lemma.

Lemma 2.4 (Lovász) *If $D_H \neq \emptyset$, then for any component T of the subgraph $G[D_H]$, and any H-optimal subgraph F of T, we have $\delta_{H'}(F) = 1$.*

This paper concerns H_n-decompositions where H_n is defined by (1.5). For convenience, we often use another prescription H_n^* defined by

$$H_n^*(v) = H_n(v) \cup \{-1\} = \{-1, 1, 3, 5, \ldots, 2n-1, 2n\}. \tag{2.3}$$

Here is the main result of this paper.

Theorem 2.5 *Let G be a graph without odd components. If G contains no H_n-factors, then there exists a nonempty subset $S \subset V(G)$ such that the subgraph $G - S$ contains at least $2n|S| + 1$ odd components, each of which contains no H_n-factors.*

Proof. By the definition (2.3), the graph G contains no H_n^*-factors. Let (A, B, C, D) be the H_n^*-decomposition of G. We shall show that the subset A can be taken as the required S.

Since $\min H_n^* = -1$, we have $B = \emptyset$, and the inequality (2.1) reduces to

$$c(D) > 2n|A|. \tag{2.4}$$

This implies $D \neq \emptyset$. Let T be a component of the subgraph $G[D]$, and F an H_n^*-optimal subgraph of T. Since $B = \emptyset$, we see that $H_B = H_n^*$. So $\delta_{H_n^*}(F) = 1$ by Lemma 2.4. Therefore, there exists a vertex, say $v_0 \in V(T)$, such that

$$\min\{\, |d_F(v) - h| : \; h \in H_n^*(v)\} = \begin{cases} 1, & \text{if } v = v_0; \\ 0, & \text{if } v \in V(T)\backslash\{v_0\}. \end{cases} \tag{2.5}$$

On the other hand, assume that $\max I_{H_n^*}(v) \geq 2n$ for some $v \in D$. By Lemma 2.1, we have

$$\min I_{H_n^*}(v) \geq 2n.$$

It follows immediately that $v \in A$, a contradiction. Thus $\max I_{H_n^*}(v) \leq 2n - 1$, namely

$$d_F(v) \leq 2n - 1$$

for any vertex $v \in T$. Consequently, the formula (2.5) implies that the degree $d_F(v_0)$ is even, while $d_F(v)$ is odd for any $v \in V(T)\backslash\{v_0\}$. Since the sum $\sum_{v \in V(T)} d_F(v)$ is even, we deduce that T is an odd component of $G[D]$.

Assume that $A = \emptyset$. Since $B = \emptyset$, by Lemma 2.2, we see that T is an odd component of G. But G has no odd components, a contradiction. So $A \neq \emptyset$.

By Lemma 2.2 and the inequality (2.4), we have

$$2n|A| < c(D) = o(D) \leq o(C) + o(D) = o(C \cup D) = o(G - A). \tag{2.6}$$

Namely, the subgraph $G - A$ has at least $2n|A| + 1$ odd components. In view of (2.5), any component T of $G[D]$ has no H_n^*-factors. Hence T has no H_n-factors. This completes the proof. ∎

For graphs satisfying Cui-Kano's condition (1.4), we obtain the following corollary immediately.

Corollary 2.6 *Any graph G satisfying the condition* (1.4) *contains an H_n-factor.*

Write $g = |V(G)|$. Noting that the condition "G has no odd components" implies that g is even. Considering graphs G with g odd in contrast, we have the following result.

Theorem 2.7 *Let G be a connected graph with g odd. Suppose that*

$$o(G-S) \le 2n|S|, \quad \forall\, S \subseteq V(G),\ S \ne \emptyset. \tag{2.7}$$

Then either G contains an H_n-factor, or G is H_n^*-critical.

Proof. Suppose that G contains no H_n-factors. Let

$$(A, B, C, D)$$

be the H_n^*-decomposition of G. From the proof of Theorem 2.5, we see that $B = \emptyset$, and obtain the inequality (2.6). Together with the condition (2.7), we see that $A = \emptyset$. Since G is connected, we find that $C = \emptyset$ by Lemma 2.2. Hence G is H_n^*-critical. This completes the proof. ∎

We remark that the condition (1.4) is not necessary for the existence of an H_n-factor in a graph. Consider the graph

$$G = K_1 + (2n+1)K_{2n+1}$$

obtained by linking a vertex K_1 to all vertices in $2n+1$ copies of the complete graph K_{2n+1}. Denote by C_j $(1 \le j \le 2n+1)$ the j-th copy of K_{2n+1}. Let $v_j \in V(C_j)$. Let F be the factor consisting of the following $2n+2$ components:

$$C_1 - v_1,\ C_2 - v_2,\ \ldots,\ C_{2n} - v_{2n},\ C_{2n+1},\ G\,[\,v_0,\, v_1,\, \ldots,\, v_{2n}\,].$$

It is easy to verify that F is an H_n-factor. However, taking the subset S to be the single vertex v_0, we see that the condition (1.4) does not hold for G.

To end this section, we point out that the coefficient $2n$ in the condition (1.4) is a sharp bound in the sense that for any $\epsilon > 0$, there exists a graph G with a subset $S \subseteq V(G)$ satisfying

$$o(G-S) < (2n+\epsilon)\,|S|, \tag{2.8}$$

and that G contains no H_n-factors. Recall that an $[a,b]$-factor is a factor F such that $a \le d_F(v) \le b$ for all vertices v. We need Las Vergnas's theorem [7].

Theorem 2.8 (Las Vergnas) *A graph G contains a $[1,n]$-factor if and only if for all subsets $S \subseteq V(G)$, the number of isolated vertices in the subgraph $G-S$ is at most $n|S|$.*

Theorem 2.9 *For any $\epsilon > 0$, there exists a graph G with a subset $S \subseteq V(G)$ satisfying the ϵ-condition (2.8) but with no H_n-factors.*

Proof. Let m be an integer such that $m > 1/\epsilon$. Let V_m be a set of m isolated vertices, and V_{2nm+1} a set of $2nm+1$ isolated vertices. Denote by $K_{m,\,2nm+1}$ the complete bipartite graph obtained by connecting each vertex in V_m with each vertex in V_{2nm+1}. Setting $S = V_m$ in Theorem 2.8, we deduce that $K_{m,\,2nm+1}$ contains no $[1, 2n]$-factors, and thus no H_n-factors. Moreover,

$$o(K_{m,\,2nm+1} - V_m) = 2nm + 1 < (2n + \epsilon)\,|V_m|.$$

This completes the proof. ∎

3 Sufficient conditions for the existence of an H_n-factor

In this section, we present some sufficient conditions for the existence of an H_n-factor. For any vertex v of G, denote by $N_G(v)$ the set of neighbors of v.

Theorem 3.1 *Let G be a graph without odd components. If for any non-adjacent vertices u and v,*

$$|\, N_G(u) \cup N_G(v)\,| > \max\left\{ \frac{g-2}{2n} - 1,\ \frac{2g-4}{4n+1},\ \frac{g-1}{2n+1},\ 4n-3 \right\}, \tag{3.1}$$

then G contains an H_n-factor.

Proof. Suppose to the contrary that G contains no H_n-factors. By Theorem 2.5, there exists a nonempty subset $S \subset V(G)$ such that the subgraph $G - S$ has at least $2ns+1$ odd components, say, $C_1, C_2, \ldots, C_{2ns+1}$, with each C_i has no H_n-factors, where $s = |S|$. Let $c_i = |V(C_i)|$. Suppose that

$$1 \le c_1 \le c_2 \le \cdots \le c_{2ns+1}. \tag{3.2}$$

It is clear that

$$2ns + 1 \le c_1 + c_2 + \cdots + c_{2ns+1} \le g - s. \tag{3.3}$$

Therefore

$$c_1 \le \frac{g-s}{2ns+1},$$

$$c_2 \le \frac{g-s-c_1}{2ns}.$$

It follows that

$$c_1 + c_2 \le \frac{2(g-s)}{2ns+1}, \tag{3.4}$$

$$c_2 \le \frac{g-s-1}{2ns}. \tag{3.5}$$

Moreover, the inequality (3.3) implies that $s \le s_*$ where

$$s_* = \frac{g-1}{2n+1}.$$

Let $u \in V(C_1)$ and $v \in V(C_2)$. Then

$$|\, N_G(u) \cup N_G(v)\,| \le s + (c_1 - 1) + (c_2 - 1). \tag{3.6}$$

By (3.4), we find that $|\, N_G(u) \cup N_G(v)\,| \le h(s)$ where

$$h(s) = \frac{2(g-s)}{2ns+1} + s - 2.$$

Note that the second derivative $h''(s) > 0$. If $s \ge 2$, then we have

$$|\, N_G(u) \cup N_G(v)\,| \le \max\{h(2), h(s_*)\} = \max\left\{ \frac{2g-4}{4n+1},\ \frac{g-1}{2n+1} \right\},$$

contradicting to the condition (3.1). Otherwise $s = 1$. In this case, if $c_2 \le 2n-1$, then $c_1 \le 2n-1$ by (3.2). By (3.6), we have

$$|\, N_G(u) \cup N_G(v)\,| \le c_1 + c_2 - 1 \le 4n - 3,$$

contradicting to (3.1). So $c_2 \ge 2n$. It is easy to verify that any complete graph K_m with $m \ge 2n$ has an H_n-factor. Since C_2 contains no H_n-factors, we deduce that C_2 is not complete. So there exist vertices u' and v' which are not adjacent in C_2. By (3.5), we have

$$|\, N_G(u') \cup N_G(v')\,| \le s + c_2 - 2 \le \frac{g-2}{2n} - 1,$$

contradicting to (3.1). This completes the proof. ∎

Observe that when $g \ge 8n^2 + 2n + 2$, one has

$$\max\left\{ \frac{g-2}{2n} - 1,\ \frac{2g-4}{4n+1},\ \frac{g-1}{2n+1},\ 4n-3 \right\} = \frac{g-2}{2n} - 1.$$

This results in the following corollary immediately.

Corollary 3.2 *Let G be a graph without odd components. If $g \ge 8n^2 + 2n + 2$, and for any non-adjacent vertices u and v,*

$$|\, N_G(u) \cup N_G(v)\,| > \frac{g-2}{2n} - 1,$$

then G contains an H_n-factor.

Now we give another sufficient condition for the existence of an H_n-factor of a graph. A graph G is said to be k-connected if it is connected when fewer than k vertices are removed from G. Let u and v be non-adjacent vertices of G. Denote by $G+uv$ the graph obtained by adding the edge (u,v) to G.

Theorem 3.3 *Let G be a k-connected simple graph with g even. Let u and v be non-adjacent vertices of G such that*

$$|\,N_G(u)\cup N_G(v)\,| \geq g-2nk. \tag{3.7}$$

Then G has an H_n-factor if and only if the graph $G+uv$ has an H_n-factor. Moreover, the lower bound $g-2nk$ in (3.7) is best possible.

Proof. The necessity is obvious. We shall prove the sufficiency. Suppose to the contrary that G has no H_n-factors. Since G is a connected graph with g even, we deduce that G has no odd components. By Theorem 2.5, there exists a nonempty subset $S\subset V(G)$ such that

$$o(G-S)\geq 2ns+1, \tag{3.8}$$

where $s=|S|$. Since G is k-connected, it is easy to see that $s\geq k$. Let $C_1,C_2,\ldots,C_q$ be the components of $G-S$, and $c_i=|V(C_i)|$. Then $q\geq 2n+1$ by (3.8). By Theorem 2.5, we can suppose that c_i is odd and C_i has no H_n-factors for any $1\leq i\leq 2ns+1$.

Let u and v be non-adjacent vertices. We have three cases.

(i) u and v belong to the same component of $G-S$. In this case, we can suppose that $u,v\in C_i$ for some $1\leq i\leq q$. By (3.8), we have

$$|\,N_G(u)\cup N_G(v)\,|\leq s+(c_i-2)=g-\sum_{j\neq i}c_j-2\leq g-2ns-2\leq g-2nk-2.$$

(ii) u and v belong to distinct components of $G-S$. In this case, we can suppose that $u\in C_i$ and $v\in C_j$ where $1\leq i<j\leq q$. By (3.8), we have

$$|N_G(u)\cup N_G(v)|\leq s+c_i+c_j-2=g-\sum_{h\notin\{i,j\}}c_h-2\leq g-2nk-1.$$

(iii) One of u and v belongs to the set S. Let F be an H_n-factor of $G+uv$. Let m be the total degree of vertices of S in F, i.e.,

$$m=\sum_{v\in S}d_F(v).$$

Since the component C_i contains no H_n-factors, and the edge (u,v) is not contained in $G-S$, there exists an edge of F connecting one vertex in C_i and another vertex in S. Therefore, each C_i corresponds an edge with one end in S. It follows that $m\geq 2ns+1$. On the other hand, each vertex in S has degree at most $2n$ in F. So $m\leq 2ns$, a contradiction. This proves the sufficiency.

Now we shall prove that the bound $g - 2nk$ is best possible. Let k be an odd number, and

$$G = K_k + (2nk + 1)K_1.$$

Then $g = 2nk + k + 1$ is even. It is easy to check that G is k-connected and

$$|N_G(u) \cup N_G(v)| = g - 2nk - 1$$

for any non-adjacent vertices $u, v \in V(G)$. It suffices to show that $G + uv$ contains an H_n-factor while G does not.

Denote by $u_1, u_2, \ldots, u_k$ the vertices of the subgraph K_k, and by $v_1, v_2, \ldots, v_{2nk+1}$ the remaining vertices in G. Suppose to the contrary that G has an H_n-factor F. Then the degree of each vertex v_i is at least 1 in F. Note that the neighbor of v_i must be some u_j. So there exists some u_j of degree at least $2n + 1$ in F. It follows that the degree of u_j does not belong to the set H_n, a contradiction. Hence G has no H_n-factors.

Now we shall show that $G + uv$ contains an H_n-factor for any non-adjacent vertices u and v. In fact, since each vertex u_i is saturated, we can suppose without loss of generality that the non-adjacent vertex pair (u, v) is taken to be (v_{2nk}, v_{2nk+1}). A factor F of $G + uv$ consists of the edge uv and the following k components:

$$\begin{aligned}
&G[u_1,\ v_1,\ v_2,\ \ldots,\ v_{2n}],\\
&G[u_2,\ v_{2n+1},\ v_{2n+2},\ \ldots,\ v_{4n}],\\
&\quad\vdots\\
&G[u_k,\ v_{2n(k-1)+1},\ v_{2n(k-1)+2},\ \ldots,\ v_{2nk-1}].
\end{aligned}$$

It is straightforward to verify that F is an H_n-factor. This completes the proof. ∎

Let H be an allowed prescription. In [9, Lemma (3.5)], Lovász gave the following result describing the H-decomposition of a graph when a vertex in A_H is removed.

Lemma 3.4 (Lovász) *Let (A, B, C, D) be the H-decomposition of G. Let v be a vertex in A, and (A', B', C', D') the H-decomposition of the subgraph $G - v$. Then*

$$A' = A - v, \quad B' = B, \quad C' = C, \quad D' = D.$$

Theorem 3.5 *Let G be a graph without odd components. Then G contains an H_n-factor if the subgraph $G - v$ contains an H_n-factor for all vertices v.*

Proof. Suppose that G has no H_n-factors. Let (A, B, C, D) be the H_n^*-decomposition of G. From the proof of Theorem 2.5, we see that $B = \emptyset$ and

$$2n|A| < c(D). \tag{3.9}$$

Moreover, every component of $G[D]$ is odd.

Assume that $A \neq \emptyset$. Let $v \in A$, and (A', B', C', D') be the H_n^*-decomposition of $G - v$. By Lemma 2.3, we have

$$c(D') \leq 2n|A'|. \tag{3.10}$$

By (3.9), (3.10) and Lemma 3.4, we deduce that

$$2n|A| \leq c(D) - 1 = c(D') - 1 \leq 2n|A'| - 1 = 2n(\,|A| - 1\,) - 1,$$

a contradiction. So $A = \emptyset$. By Lemma 2.2, any component of $G[D]$ is an odd component of G. But G has no odd components, a contradiction. This completes the proof. ∎

References

[1] J. Akiyama and M. Kano, Factors and Factorizations of Graphs, Lecture Notes in Math., vol. 2031, Springer, 2011.

[2] A. Amahashi, On factors with all degree odd, Graphs Combin. 1 (1985), 111–114.

[3] G. Cornuéjols, General factors of graphs, J. Combin. Theory Ser. B 45 (1988), 185–198.

[4] Y. Cui and M. Kano, Some results on odd factors of graphs, J. Graph Theory 12 (1988), 327–333.

[5] M. Kano, G. Y. Katona, and J. Szabó, Elementary graphs with respect to f-parity factors, Graphs Combin. 25 (2009), 717–726.

[6] M. Kouider and P. D. Vestergaard, Connected factors in graphs–a survey, Graphs Combin. 21 (2005), 1–26.

[7] M. Las Vergnas, An extension of Tutte's 1-factor theorem, Discrete Math. 23 (1978), 241–255.

[8] L. Lovász, The factorization of graphs, in: Combinatorial Structures and Their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969), 243–246, Gordon and Breach, New York, 1970.

[9] L. Lovász, The factorization of graphs. II, Acta Math. Hungar. 23 (1972), 223–246.

[10] M. D. Plummer, Graph factors and factorization: 1985–2003: A survey, Discrete Math. 307 (2007), 791–821.

[11] J. Topp and P. D. Vestergaard, Odd factors of a graph, Graphs Combin. 9 (1993), 371–381.

[12] R. Q. Yu and G. Liu, Graph Factors and Matching Extensions, Springer, 2009.